Calculating π with Inscribed Polygons

Introduction

What is π?

The constant value of π has proved to be useful in a wide array of mathematical fields. It began as a placeholder for a number that ancient mathematicians felt must exist: The ratio of the measure of the perimeter of a circle (called the circumference) to its diameter. We can see this right away if we realize that π is the Greek letter corresponding to our letter p, the first letter of "perimeter". Ancient mathematicians reasoned that the circumference of a circle would equal its diameter times this constant value, which came to be called π. The diameter of a circle (the greatest distance between two sides) is twice its radius (the distance from the center to the side). That is:

$$c = \pi d = 2\pi r \tag{1}$$

Since the diameter, radius, and circumference of a circle could all be directly measured from specific circles (such as wheels), that meant that π could be estimated from the other values. However, because real world objects are generally imperfect, π determined this way would always be an approximation, never an exact value.

Ancient mathematicians also thought that π would be some sort of decimal, or rational, number. They thought it had to be possible to "square the circle", that is, to find some number that would allow people to find a square and a circle that had exactly the same area. It turns out that this is not possible, because π is an **irrational number**: Expressed as a decimal, its digits never repeat.

> ***Irrational number:*** *A number, such as π and $\sqrt{2}$, that cannot be expressed exactly in terms of a fraction.*

This makes π impossible to represent exactly in decimal or fractional form; 3.14159 and 22/7 are both close, but they're not exact. This may lead curious mathematics students to the question: How was π calculated in the first place?

Today, there are some fairly sophisticated ways of determining π using calculus. However, in the Third Century, the Chinese mathematician Liu Hui came up with a pretty close approximation of π using simple geometry and a whole lot of multiplication.

Step 1: An Inscribed Hexagon

First of all, Liu Hui reasoned that the circumference of a circle had to be greater than the perimeter of a hexagon that's contained fully within the circle. Specifically, he looked at an **inscribed** hexagon, that is, the largest regular hexagon that can fit inside a circle.

> ***Inscribed polygon:*** *The largest polygon which fits entirely into a circle or another polygon.*

Why did he choose a hexagon? Any regular polygon (an equilateral triangle, a square, a pentagon, and so on) would have worked, but the mathematics is easiest using either a square or a hexagon. For an inscribed hexagon, the first step is very simple, because a hexagon consists of six equilateral triangles. For a circle with a diameter of 2, each triangle has sides of 1. Look at figure 1. Each of the triangles is equilateral. For each one, two of the sides are same length as the radius of the circle. Since the diameter of the circle is 2, its radius is half that, or 1. Hence each triangle has sides of length 1. This means that the perimeter of the hexagon is six times one. We'll show the number of sides of the hexagon as a subscript, that is:

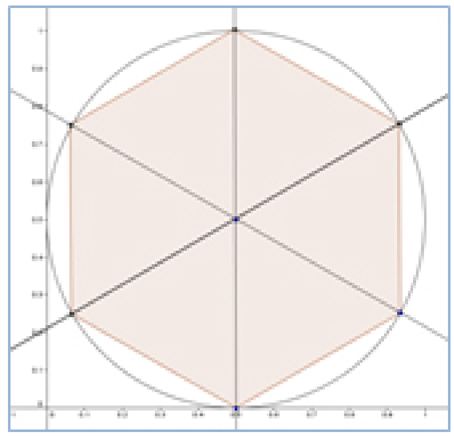

Figure 1: *An inscribed hexagon*

$$p_6 = 6 \cdot 1 = 6 \tag{2}$$

The circumference of the circle has to be greater than this, so π has to be at least half of this:

$$\pi > \frac{p_6}{2} = \frac{6}{2} = 3 \tag{3}$$

So far, so good. Based on what we already know about π, we appear to be on the right track.

Step 2: An Inscribed Dodecagon

A dodecagon has twelve sides, twice the number of sides as a hexagon. The next realization that Liu Hui had was that if the perimeter of an inscribed hexagon was close in length to the circumference of a circle, the perimeter of an inscribed dodecagon would be even closer.

If you consider figure 2, you can see why. The white areas represent the difference between the dodecagon and the circle, much less than the lighter shaded areas that represent the difference between the hexagon and the dodecagon.

We would therefore expect the perimeter of the dodecagon to give us a much better approximation for π than the hexagon did.

However, this requires some more sophisticated geometry. Look figure 3; I've zoomed in on the lower right portion of the dodecagon. The goal is to calculate the length of $\overline{BE}$. The formula for the total perimeter of the dodecagon is:

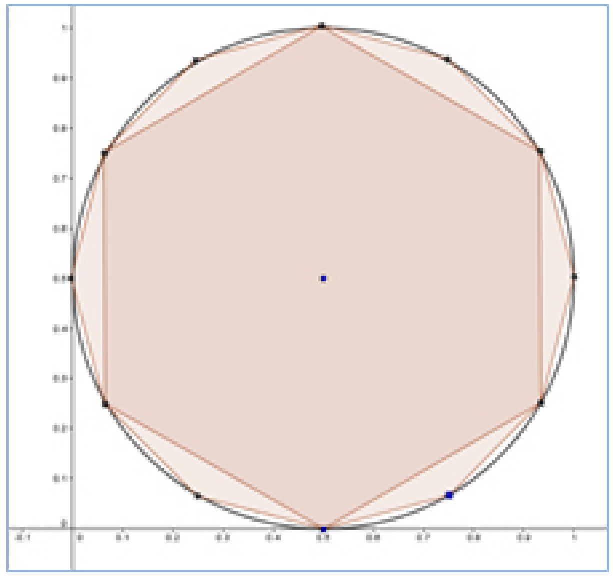

Figure 2: *Inscribing a hexagon and dodecagon*

$$p_1 2 = 12 \cdot m\overline{BE} \tag{4}$$

We know that $m\overline{AE} = m\overline{AB} = 1$, because these segments represent radii of the circle.

Because $m\overline{DE} = m\overline{BE}$, we can easily demonstrate that $\overline{AE}$ bisects $\angle BAD$. Since $m\overline{BD} = 1$ and $m\overline{AE} = 1$ bisects $\angle BAD$, we know that $m\overline{BD} = 1/2$.

Since $m\overline{AD} = m\overline{AB}$, $m\overline{AC} = m\overline{AC}$, and $m\overline{DC} = m\overline{CB}$, it must be the case (by the SSS Postulate) that $\triangle ADC$ and $\triangle ABC$ are congruent. Since $m\angle DCB = 180°$, $m\angle ACB = m\angle ACD = 180°/2 = 90°$, making $\triangle ABC$ a right triangle.

From here, we can use the Pythagorean theorem to calculate $m\overline{AC}$:

$$m\overline{AC}^2 = m\overline{AB}^2 - m\overline{BC}^2 = 1^2 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} \tag{5a}$$

$$m\overline{AC} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} \tag{5b}$$

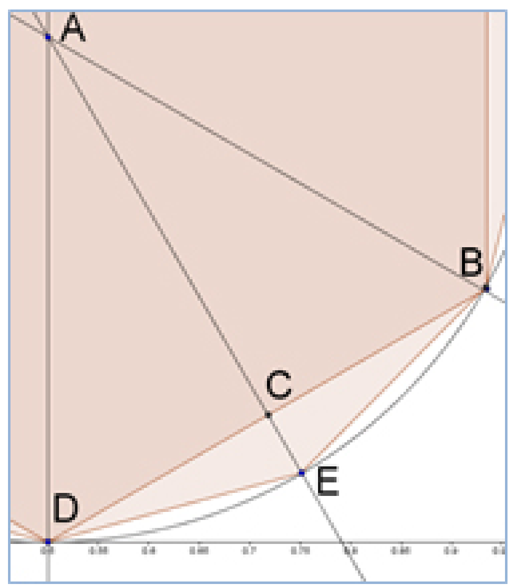

Figure 3: *Detail of figure 2*

Recall that our goal is to calculate the length of $m\overline{BE}$. We already know $m\overline{BC}$ and that $m\angle ACB = m\angle BCE = 90°$. That means we need one more piece to use the Pythagorean theorem: The length of $m\overline{CE}$. We know $m\overline{AC}$ and $m\overline{AE}$, so we can calculate $m\overline{CE}$:

$$m\overline{CE} = m\overline{AE} - m\overline{AC} \tag{6a}$$

$$= 1 - \frac{\sqrt{3}}{2} \tag{6b}$$

$$= \frac{2}{2} - \frac{\sqrt{3}}{2} \tag{6c}$$

$$= \frac{2-\sqrt{3}}{2} \tag{6d}$$

Finally, we can calculate the length of $m\overline{BE}$ using the Pythagorean theorem:

$$m\overline{BE}^2 = m\overline{BC}^2 + m\overline{CE}^2 \tag{7a}$$

$$= \frac{1}{2}^2 + \left(\frac{2-\sqrt{3}}{2}\right)^2 \tag{7b}$$

$$= \frac{1}{4} + \frac{(2-\sqrt{3})^2}{4} \tag{7c}$$

$$= \frac{1}{4} + \frac{2^2 - 2 \cdot 2\sqrt{3} + \sqrt{3}^2}{4} \tag{7d}$$

$$= \frac{1 + 4 - 4\sqrt{3} + 3}{4} \tag{7e}$$

$$= \frac{8 - 4\sqrt{3}}{4} \tag{7f}$$

$$= 2 - \sqrt{3} \tag{7g}$$

$$m\overline{BE} = \sqrt{2-\sqrt{3}} \tag{7h}$$

$$\approx 0.517638 \tag{7i}$$

This gives us $p_1 2 \approx 6.211657$ and $\pi > 3.105829$. Notice that we are now quite a bit closer to what we know to be π

A Note on Interpolation

> ***Interpolation:*** *A method of determining an unknown value by finding known values above and below it.*

Keep in mind that Liu Hui and other mathematicians of the era didn't have calculators (other than the abacus); they had to do all of this mathematics by hand. If you've never calculated square roots by hand, the simplest process is to use *interpolation*. Interpolation involves finding two values close to what you want, then guessing about a value in between those two values. For instance, let's say we want to figure out what the square root of 3 is. We know that $\sqrt{1} = 1$ and $\sqrt{4} = 2$, so $\sqrt{3}$ has to be between those two numbers.

Let's try 1.6, 1.7, and 1.8:

$$1.6 \cdot 1.6 = 2.56 \tag{8}$$

$$1.7 \cdot 1.7 = 2.89 \tag{9}$$

$$1.8 \cdot 1.8 = 3.24 \tag{10}$$

Therefore, we know that $1.7 < \sqrt{3} < 1.8$. We could then try 1.72, 1.73, and 1.74:

$$1.72 \cdot 1.72 = 2.9584 \tag{11}$$

$$1.73 \cdot 1.73 = 2.9929 \tag{12}$$

$$1.74 \cdot 1.74 = 3.0276 \tag{13}$$

Now we know that $1.73 < \sqrt{3} < 1.74$. We can continue in the same way to find additional places. Because $\sqrt{3}$ is irrational, we will never find an exact value, but each digit will get us closer and closer.

Step 3: Inscribing Other Polygons

The next step is to determine the perimeter of an icosikaitetragon,[1] a 24-sided polygon. You can see from figure 4, which is a close-up of only a section of the circle, how close we're getting this time (the dots don't completely line up because of limitations of GeoGebra, the software I'm using). The white represents the gap between the circle and the icosikaitetragon, the lightest shade the gap between

[1] Hardly anybody actually uses the Greek terms for polygons with more than 12 sides. After dodecagon, most mathematicians just say "13-gon", "14-gon", and so on. So don't worry about these big words; I'm including them mostly for their amusement value. For more information, visit Name That Polygon (`http://www.diwa.ph/global/UserFiles/epages/1216810772764tsk1413_namepoly.pdf`; downloaded 11/23/11).

that and the dodecagon, and the middle shade the gap between the dodecagon and the hexagon. With each successive step, we're getting closer and closer to having no gap at all with the circle.

We use the same basic process: We figure out the midpoint of one of the sides of the dodecagon (12-sided polygon). We use geometry to calculate the lengths of the sides of the new, now even smaller, triangles. We then calculate the length of each side of the new polygon.

As you might imagine, the calculations are even trickier than last time. You can work it out if you'd like the exercise, but at the end of it we have this as the measure of each side:

$$\sqrt{2-\sqrt{2+\sqrt{3}}} \approx 0.261052 \tag{14}$$

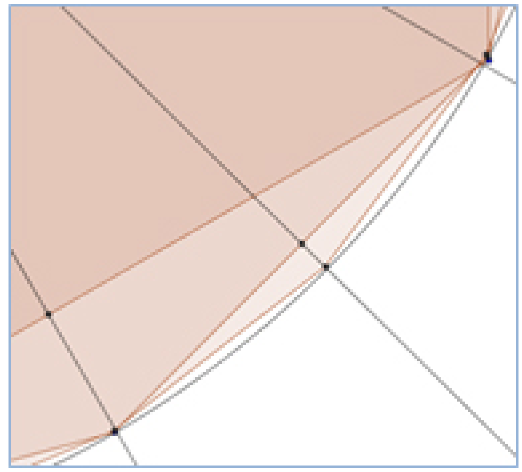

Figure 4: *Detail of an inscribed icosikaitetragon*

Because this is a 24-sided polygon, π is a little more than 12 times this, which is 3.132629.

If we inscribe a regular tetracontakaioctagon, a 48-sided polygon, we get this measure for each side:

$$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} \approx 0.130806 \tag{15}$$

Because this is a 48-sided polygon, π is a little more than 24 times this, which is 3.139350.

If we inscribed an enneacontakaihexagon, a 96-sided polygon, we would get this measure for each side:

$$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} \approx 0.065438 \tag{16}$$

Because this is a 96-sided polygon, π is a little more than 48 times this, which is 3.141032.

At this point, you might be able to notice a pattern. You no longer need to use the geometry of the inscribed polygons: You just need to follow the pattern and keep track of the number of times you repeat it. Notice that each time we double the number of sides, we insert another square root level with "2+" inside of it. For the 198 sided hectaenneacontakaihexagon, the formula for estimating π is:

$$\pi \approx 96 \cdot \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}} \approx 3.141452 \tag{17}$$

And so on. Notice that, with each iteration, we are getting closer to what we use as π. The table shows a listing of the first twelve approximations of π based on the inscribed hexagon method.

Number of sides	Perimeter/2
6	3.000000
12	3.105829
24	3.132629
48	3.139350
96	3.141032
192	3.141452
384	3.141558
768	3.141584
1536	3.141590
3072	3.141592
6144	3.141593
12288	3.141593

Table 1: *π approximations by sides.*

By the way, if we had started with an inscribed square instead of an inscribed hexagon, our formula would differ only by the last value. For instance, the calculation for a polygon with 256 sides (a dohectapentacontahexagon) is:

$$\pi \approx 128 \cdot \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}} \approx 3.141277 \quad (18)$$

Notice that the most embedded term is $\sqrt{2}$ instead of $\sqrt{3}$.

Conclusion

While calculus as we know it today was not going be invented for over a millennium, this method for approximating π relied on a concept that would be crucial for calculus: The idea of the mathematical *limit*. Put in terms of limits, the method of inscribing polygons to approximate π can be written this way:

> ***Limit:*** *A mathematical process for determining what value a function or other process is converging upon.*

$$\lim_{s \to \infty} p_s = 2\pi \quad (19)$$

where p_s is the perimeter of regular polygon with s sides. This formula is a rigorous mathematical way of saying, “As you increase the number of sides of a polygon inscribed within a circle of radius 1, its perimeter gets closer and closer to twice the value of π. If you could go on forever, you would eventually get to twice π exactly."

There are now more sophisticated methods for calculating π to increasingly accurate decimal points. For the advanced student of mathematics, Wolfram Alpha provides a variety of infinite sums and integrals which represent π.[2]

However, Liu Hui got impressively close to the value we now use with only geometry and a whole lot of mathematical elbow grease.

[2] `http://www.wolframalpha.com/input/?i=how+to+calculate+pi`, accessed 11/20/11